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The averaged Hamiltonian of a time-independent nearly multiple-periodic system

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Abstract. A discussion is presented of the canonical averaging of time-independent, nearly multiple-periodic systems having Hamiltonians of the form

$$H = H_0(J_\alpha)+\lambda H_1(w_\alpha, J_\alpha, q_k, p_k)+\lambda^2 H_2(w_\alpha, J_\alpha, q_k, p_k)+$$

where $H_1, H_2, \ldots$ are periodic functions of the angles w_α. A perturbation procedure is given for constructing a direct canonical transformation converting the Hamiltonian into a new one, independent of the proper angles w_α, irrespective of whether the system is non-degenerate or intrinsically degenerate. In each case constants of motion to all orders of the perturbation theory exist, corresponding to each proper angle w_α

1. Introduction

The method of averaging has a large variety of modifications, each adapted to a particular class of dynamical systems (Mitropolskii 1971, Giacaglia 1972). In this article we are concerned with time-independent Hamiltonian systems in which we want to separate rapid oscillatory motions from slow secular evolution of the whole system. Mitropolskii and Zubarev were the first to deal with this sort of problem by means of the so called method of rapidly rotating phase (Bogolyubov and Mitropolskii 1955). Coffey and Ford (1969) generalized this method to systems with several rapid phases and discussed its connection with the classical canonical perturbation theory of Poincaré (Born 1960, Fues 1927). If applied to a canonical set of equations, the method of Coffey and Ford yields equations which are not necessarily canonical. An integral of motion, the Hamiltonian, may therefore be lost. In an attempt to make the method canonical, Coffey (1969) examined nearly multiple-periodic systems having Hamiltonians of the form

$$H = H_0(J_\alpha)+\lambda H_1(w_\alpha, J_\alpha; q_k, p_k) \tag{1.1}$$

where H_1 is a periodic function of the angles w_α. Coffey constructed a transformation averaging the corresponding canonical equations over the angles w_α, but succeeded in making the transformation a canonical one only in the case where the system is non-degenerate. The present author (Sedláček 1974) modified therefore the classical perturbation theory of Poincaré which is canonical by its very nature, so as to derive, not averaged canonical equations as Coffey did, but rather an averaged Hamilton–Jacobi equation for the generator of the canonical transformation that converts the original

Hamiltonian into one involving only momenta. This was achieved for both the non-degenerate and intrinsically degenerate systems (for classification of degeneracies see Born 1960, Fues 1927). Although the application of this method furnished interesting results (Sedláček 1974), there are some disadvantages connected with such an approach. First, the canonical transformation found is implicit, defined in terms of its generator and, secondly, the Hamilton–Jacobi equation, though providing a complete description of the problem and capable of serving as an integral of motion, does not permit the introduction of averaged variables and their corresponding equations of motion.

The aim of the present paper is to show that a minor modification of our previous scheme enables one to remove both drawbacks at a time. Only non-degenerate systems are treated, since intrinsically degenerate systems with Hamiltonians of the form (1.1) are reducible to non-degenerate ones by introduction of improper angles (Sedláček 1975). In § 2 we define a general near-identity canonical transformation by its generating function developed into a power series of the perturbation parameter λ and, by inverting the transformation relations perturbationally, arrive at a direct canonical transformation dependent on an infinite number of arbitrary functions. In § 3 the Hamiltonian of a nearly multiple-periodic system is transformed by this transformation and the arbitrary functions are determined successively from the requirement that the transformation be periodic (ie secularity-free) in the angle-like variables w_α and the new Hamiltonian be independent of w_α. The canonical transformation and the averaged Hamiltonian are thus obtained by a single procedure without any appeal to either the Hamiltonian–Jacobi equation or the canonical equations of motion.

In constructing a general direct near-identity canonical transformation we have followed the most elementary procedure due to Burshtein and Solovev (1961), although a powerful method based on the apparatus of Lie's series is available (Hori 1966, Stern 1970). The method of constructing the averaged Hamiltonian is, however, different from each of the schemes of Fedortshenko (1957), Burshtein and Solovev (1961) and Sturrock (1964) which are all concerned with averaging of time-dependent Hamiltonians over the time. The reader will find a comprehensive example of the present averaging scheme in a paper on nonlinear instability of three resonantly interacting electrostatic waves on two counter-streaming electron beams (Sedláček 1974), submitted to *Journal of Plasma Physics*.

2. Direct near-identity canonical transformation

Let a near-identity canonical transformation from the set of old canonically conjugate variables q_k, p_k $(k = 1, 2, \ldots, n)$ to a new set of canonically conjugate variables Q_k, P_k be generated by a function $S(q_k, P_k)$, developed into a power series of a small parameter λ,

$$S(q_k, P_k) = \sum_{k=1}^{n} q_k P_k + \lambda \mathscr{S}_1(q_k, P_k) + \lambda^2 \mathscr{S}_2(q_k, P_k) + \ldots, \tag{2.1}$$

the functions $\mathscr{S}_1, \mathscr{S}_1, \ldots$ being arbitrary. The corresponding transformation relations

$$Q_k = \frac{\partial S}{\partial P_k} = q_k + \lambda \frac{\partial \mathscr{S}_1}{\partial P_k} + \lambda^2 \frac{\partial \mathscr{S}_2}{\partial P_k} + \ldots \tag{2.2}$$

$$p_k = \frac{\partial S}{\partial q_k} = P_k + \lambda \frac{\partial \mathscr{S}_1}{\partial q_k} + \lambda^2 \frac{\partial \mathscr{S}_2}{\partial q_k} + \ldots \tag{2.3}$$

are implicit. We seek an inversion of these relations in the explicit form of a direct canonical transformation from the old to the new variables:

$$q_k = Q_k + \lambda\, \mathcal{Q}_{1k}(Q_l, P_l) + \lambda^2\, \mathcal{Q}_{2k}(Q_l, P_l) + \dots \tag{2.4}$$

$$p_k = P_k + \lambda \mathcal{P}_{1k}(Q_l, P_l) + \lambda^2 \mathcal{P}_{2k}(Q_l, P_l) + \dots . \tag{2.5}$$

To obtain the unknown functions $\mathcal{Q}_{1k}$, $\mathcal{Q}_{2k}, \dots ; \mathcal{P}_{1k}, \mathcal{P}_{2k}, \dots$ we substitute the foregoing relations into the equations (2.2), (2.3), expand into powers of λ and equate the coefficients of the same powers of λ. We thus obtain a set of equations which is easily solved recursively to any desired order. The first three steps yield (the arguments of all functions are Q_k, P_k)

$$\mathcal{Q}_{1k} = -\frac{\partial \mathcal{S}_1}{\partial P_k} \tag{2.6a}$$

$$\mathcal{Q}_{2k} = -\frac{\partial \mathcal{S}_2}{\partial P_k} + \sum_{l=1}^{n} \frac{\partial^2 \mathcal{S}_1}{\partial P_k \partial Q_l} \frac{\partial \mathcal{S}_1}{\partial P_l} \tag{2.6b}$$

$$\begin{aligned}\mathcal{Q}_{3k} = & -\frac{\partial \mathcal{S}_3}{\partial P_k} + \sum_{l=1}^{n} \frac{\partial^2 \mathcal{S}_2}{\partial P_k \partial Q_l} \frac{\partial \mathcal{S}_1}{\partial Q_l} + \sum_{l=1}^{n} \frac{\partial^2 \mathcal{S}_1}{\partial P_k \partial Q_l} \frac{\partial \mathcal{S}_2}{\partial P_l} \\ & - \sum_{l_1=1}^{n} \sum_{l_2=1}^{n} \frac{\partial^2 \mathcal{S}_1}{\partial P_k \partial Q_{l_1}} \frac{\partial^2 \mathcal{S}_1}{\partial P_k \partial Q_{l_2}} \frac{\partial \mathcal{S}_1}{\partial P_{l_2}} - \frac{1}{2} \sum_{l_1=1}^{n} \sum_{l_2=1}^{n} \frac{\partial^3 \mathcal{S}_1}{\partial P_k \partial Q_{l_1} \partial Q_{l_2}} \frac{\partial \mathcal{S}_1}{\partial P_{l_1}} \frac{\partial \mathcal{S}_1}{\partial P_{l_2}}\end{aligned} \tag{2.6c}$$

$$\mathcal{P}_{1k} = \frac{\partial \mathcal{S}_1}{\partial Q_k} \tag{2.7a}$$

$$\mathcal{P}_{2k} = \frac{\partial \mathcal{S}_2}{\partial Q_k} - \sum_{l=1}^{n} \frac{\partial^2 \mathcal{S}_1}{\partial Q_k \partial Q_l} \frac{\partial \mathcal{S}_1}{\partial P_l} \tag{2.7b}$$

$$\begin{aligned}\mathcal{P}_{3k} = & \frac{\partial \mathcal{S}_3}{\partial Q_k} - \sum_{l=1}^{n} \frac{\partial^2 \mathcal{S}_2}{\partial Q_k \partial Q_l} \frac{\partial \mathcal{S}_1}{\partial P_l} - \sum_{l=1}^{n} \frac{\partial^2 \mathcal{S}_1}{\partial Q_k \partial Q_l} \frac{\partial \mathcal{S}_2}{\partial P_l} \\ & + \sum_{l_1=1}^{n} \sum_{l_2=1}^{n} \frac{\partial^2 \mathcal{S}_1}{\partial Q_k \partial Q_{l_1}} \frac{\partial \mathcal{S}_1}{\partial P_{l_1} \partial Q_{l_2}} \frac{\partial \mathcal{S}_1}{\partial P_{l_2}} + \frac{1}{2} \sum_{l_1=1}^{n} \sum_{l_2=1}^{n} \frac{\partial^3 \mathcal{S}_1}{\partial Q_k \partial Q_{l_1} \partial Q_{l_2}} \frac{\partial \mathcal{S}_1}{\partial P_{l_1}} \frac{\partial \mathcal{S}_1}{\partial P_{l_2}}\end{aligned} \tag{2.7c}$$

3. Averaging process

We assume a Hamiltonian of the same form as in the previous article (Sedláček 1975):

$$H(w_\alpha^0, J_\alpha^0 ; q_k^0, p_k^0) = H_0(J_\alpha^0) + \lambda H_1(w_\alpha^0, J_\alpha^0 ; q_k^0, p_k^0) + \lambda^2 H_2(w_\alpha^0, J_\alpha^0 ; q_k^0, p_k^0) + \dots \tag{3.1}$$

in which w_α^0, J_α^0 ($\alpha = 1, 2, \dots, m$) are the angle-action variables of the multiple-periodic degrees of freedom and q_k^0, p_k^0 ($k = 1, 2, \dots, n$) are the coordinates and momenta, respectively, of those degrees of freedom that are 'frozen' (q_k^0 = constant, p_k^0 = constant) in the unperturbed state of motion ($\lambda = 0$). The functions $H_1, H_2, \dots$ are periodic in the angles w_α^0 with period 2π. The old variables will be labelled by a superscript zero, the new variables will have no superscript.

The Hamiltonian (3.1) will be transformed by means of a near-identity canonical transformation generated by the function

$$S(w_\alpha^0, J_\alpha; q_k^0, p_k) = \sum_{\alpha=1}^{m} w_\alpha^0 J_\alpha + \sum_{k=1}^{n} q_k^0 p_k + \lambda \mathscr{S}_1(w_\alpha^0, J_\alpha; q_k^0, p_k) + \lambda^2 \mathscr{S}_2(w_\alpha^0, J_\alpha; q_k^0, p_k) + \dots \quad (3.2)$$

The corresponding direct transformation relations are constructed according to the formulae (2.6), (2.7) (only those terms are written which are necessary to obtain the functions $\mathscr{S}_1$ and $\mathscr{S}_2$):

$$w_\alpha^0 = w_\alpha + \lambda \mathscr{W}_{1\alpha} + \lambda^2 \mathscr{W}_{2\alpha} + \dots \quad (3.3)$$

$$J_\alpha^0 = J_\alpha + \lambda \mathscr{J}_{1\alpha} + \lambda^2 \mathscr{J}_{2\alpha} + \dots \quad (3.4)$$

$$q_k^0 = q_k + \lambda \mathscr{Q}_{1k} + \lambda^2 \mathscr{Q}_{2k} + \dots \quad (3.5)$$

$$p_k^0 = p_k + \lambda \mathscr{P}_{1k} + \lambda^2 \mathscr{P}_{2k} + \dots \quad (3.6)$$

$$\mathscr{W}_{1\alpha} = -\frac{\partial \mathscr{S}_1}{\partial J_\alpha}, \qquad \mathscr{W}_{2\alpha} = -\frac{\partial \mathscr{S}_2}{\partial J_\alpha} + \dots \quad (3.7)$$

$$\mathscr{J}_{1\alpha} = \frac{\partial \mathscr{S}_1}{\partial w_\alpha}, \qquad \mathscr{J}_{2\alpha} = \frac{\partial \mathscr{S}_2}{\partial w_\alpha} - \dots \quad (3.8)$$

$$\mathscr{Q}_{1k} = -\frac{\partial \mathscr{S}_1}{\partial p_k}, \qquad \mathscr{Q}_{2k} = -\frac{\partial \mathscr{S}_2}{\partial p_k} + \dots \quad (3.9)$$

$$\mathscr{P}_{1k} = \frac{\partial \mathscr{S}_1}{\partial q_k}, \qquad \mathscr{P}_{2k} = \frac{\partial \mathscr{S}_2}{\partial q_k} - \dots . \quad (3.10)$$

On substitution into (3.1) and expansion into powers of λ, there results:

$$\begin{aligned} H^{\mathrm{T}}(w_\alpha, J_\alpha; q_k, p_k) &= H_0(J_\alpha) + \lambda\left(\sum_{\beta=1}^{m} \frac{\partial H_0}{\partial J_\beta}\frac{\partial \mathscr{S}_1}{\partial w_\beta} + H_1(w_\alpha, J_\alpha; q_k, p_k)\right) \\ &+ \lambda^2\left(\sum_{\beta=1}^{m} \frac{\partial H_0}{\partial J_\beta}\frac{\partial \mathscr{S}_2}{\partial w_\beta} + \frac{1}{2!}\sum_{\beta_1=1}^{m}\sum_{\beta_2=1}^{m} \frac{\partial^2 H_0(J_\alpha)}{\partial J_{\beta_1}\partial J_{\beta_2}}\frac{\partial \mathscr{S}_1}{\partial w_{\beta_1}}\frac{\partial \mathscr{S}_1}{\partial w_{\beta_2}} \right. \\ &- \sum_{\beta=1}^{m} \frac{\partial H_1}{\partial w_\beta}\frac{\partial \mathscr{S}_1}{\partial J_\beta} + \sum_{\beta=1}^{m} \frac{\partial H_1}{\partial J_\beta}\frac{\partial \mathscr{S}_1}{\partial w_\beta} - \sum_{l=1}^{n} \frac{\partial H_1}{\partial q_l}\frac{\partial \mathscr{S}_1}{\partial p_l} \\ &\left. + \sum_{l=1}^{n} \frac{\partial H_1}{\partial p_l}\frac{\partial \mathscr{S}_1}{\partial q_l} + H_2(w_\alpha, J_\alpha; q_k, p_k)\right) + \dots . \end{aligned} \quad (3.11)$$

To specify the arbitrary functions $\mathscr{S}_1, \mathscr{S}_2, \dots$ and thereby the new Hamiltonian H^{T}, an averaging process with respect to the angle-like variables w_α will be performed. A periodic function $F(w_\alpha)$ is decomposed into the mean (average) value $\langle F(w_\alpha)\rangle$, defined by

$$\langle F(w_\alpha)\rangle = \int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi} F(w_\alpha)\, \mathrm{d}w_1\, \mathrm{d}w_2 \dots \mathrm{d}w_m, \quad (3.12)$$

and the periodic part

$$\{F(w_\alpha)\} = F(w_\alpha) - \langle F(w_\alpha)\rangle. \qquad (3.13)$$

The functions $\mathscr{S}_1, \mathscr{S}_2, \ldots$ are determined so as to be periodic functions of the angle-like variables:

$$\sum_{\beta=1}^{m} \frac{\partial H_0}{\partial J_\beta}\frac{\partial \mathscr{S}_1}{\partial w_\beta} + \{H_1(w_\alpha, J_\alpha; q_k, p_k)\} = 0 \qquad (3.14a)$$

$$\sum_{\beta=1}^{m} \frac{\partial H_0}{\partial J_\beta}\frac{\partial \mathscr{S}_2}{\partial w_\beta} + \left\{\frac{1}{2!}\sum_{\beta_1=1}^{m}\sum_{\beta_2=1}^{m} \frac{\partial^2 H_0}{\partial J_{\beta_1}\partial J_{\beta_2}}\frac{\partial \mathscr{S}_1}{\partial w_{\beta_1}}\frac{\partial \mathscr{S}_2}{\partial w_{\beta_2}} - \sum_{\beta=1}^{m}\frac{\partial H_1}{\partial w_\beta}\frac{\partial \mathscr{S}_1}{\partial J_\beta} + \sum_{\beta=1}^{m}\frac{\partial H_1}{\partial J_\beta}\frac{\partial \mathscr{S}_1}{\partial w_\beta} - \sum_{l=1}^{n}\frac{\partial H_1}{\partial q_l}\frac{\partial \mathscr{S}_1}{\partial p_l} + \sum_{l=1}^{n}\frac{\partial H_1}{\partial p_l}\frac{\partial \mathscr{S}_1}{\partial q_l} + H_2(w_\alpha, J_\alpha; q_k, p_k)\right\} = 0 \qquad (3.14b)$$

$$\vdots$$

which requires that the new Hamiltonian be an average of the right-hand side of equation (3.11) over the angle-like variables:

$$\bar{H}(J_\alpha; q_k, p_k) = \bar{H}_0(J_\alpha) + \lambda\bar{H}_1(J_\alpha; q_k, p_k) + \lambda^2\bar{H}_2(J_\alpha; q_k, p_k) + \ldots \qquad (3.15)$$

with

$$\bar{H}_0 = H_0(J_\alpha) \qquad (3.16a)$$

$$\bar{H}_1 = \langle H_1(w_\alpha, J_\alpha; q_k, p_k)\rangle \qquad (3.16b)$$

$$\bar{H}_2 = \left\langle \frac{1}{2!}\sum_{\beta_1=1}^{m}\sum_{\beta_2=1}^{m} \frac{\partial^2 H_0}{\partial J_{\beta_1}\partial J_{\beta_2}}\frac{\partial \mathscr{S}_1}{\partial w_{\beta_1}}\frac{\partial \mathscr{S}_1}{\partial w_{\beta_2}} - \sum_{\beta=1}^{m}\frac{\partial H_1}{\partial w_\beta}\frac{\partial \mathscr{S}_1}{\partial J_\beta} + \sum_{\beta=1}^{m}\frac{\partial H_1}{\partial J_\beta}\frac{\partial \mathscr{S}_1}{\partial w_\beta} - \sum_{l=1}^{n}\frac{\partial H_1}{\partial q_l}\frac{\partial \mathscr{S}_1}{\partial p_l} + \sum_{l=1}^{n}\frac{\partial H_1}{\partial p_l}\frac{\partial \mathscr{S}_1}{\partial q_l} + H_2(w_\alpha, J_\alpha; q_k, p_k)\right\rangle \qquad (3.16c)$$

$$\vdots$$

It is seen that the calculational procedure is very much the same as in the previous article: first the function $\mathscr{S}_1$, $\mathscr{S}_2, \ldots$ are calculated from equations (3.14*a*), (3.14*b*), ... in the form of multiple Fourier series of the angle-like variables w_α, the coefficients of which depend on J_α, q_k, p_k. The functions $\mathscr{S}_1, \mathscr{S}_2; \ldots$ are then substituted into equations (3.16) which give the new Hamiltonian $\bar{H}$, independent of the angle-like variables w_α. It is worth noticing that only the Hamiltonian $\bar{H}_1$ is a simple average of H_1; every higher averaged Hamiltonian $\bar{H}_n$ contains additional terms composed of $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_{n-1}$ which, as seen from equations (3.14), may be nonzero even if all the perturbational Hamiltonians $H_2, H_3, \ldots, H_n$ are vanishing†.

The new (averaged) variables $w_\alpha, J_\alpha, q_k, p_k$ obey the following canonical equations of motion:

$$\dot{w}_\alpha = \frac{\partial \bar{H}}{\partial J_\alpha}, \qquad \dot{J}_\alpha = 0 \qquad (3.17)$$

† This fact will be shown to be essential in demonstrating the limitation of a nonlinear instability of resonantly interacting waves (Sedláček 1974): the Hamiltonian $\bar{H}_1$ causes an instability which is limited by $\bar{H}_2$ although $H_2, H_3, \ldots$ are all absent.

$$\dot{q}_k = \frac{\partial \bar{H}}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial \bar{H}}{\partial q_k}. \tag{3.18}$$

All the new variables J_α are constants to all orders of the perturbation theory. In each of equations (3.17), (3.18) these variables may therefore be regarded as mere parameters so that only equations (3.18) for q_k, p_k are to be solved, independently of equations (3.17). This solution is then substituted into the right-hand sides of the first equations of the pairs (3.17) from which, by simple integration in time, w_α are found. The angular degrees of freedom are thus effectively eliminated and the problem is reduced to the solution of the set of equations (3.18) for the slow evolution of the 'frozen' variables. This is to be compared with the result of the previous article where the problem was reduced to the solution of an averaged Hamilton–Jacobi equation also involving only the 'frozen' variables (equation (2.12) of that article). The same effective reduction of the number of degrees of freedom is thus achieved in both cases.

The set (3.18) can also be solved with the aid of its associated Hamilton–Jacobi equation which follows from the averaged Hamiltonian (3.15). The separation of variables, if feasible, may considerably facilitate the solution. There is, however, a difference between this Hamilton–Jacobi equation and the averaged Hamilton–Jacobi equation derived in the previous article because in the latter it is the old (unaveraged) coordinates q_k^0 that appear as independent variables.

References

Bogolyubov N N and Mitropolskii Yu A 1955 *Asimptotitsheskie Metody v Teorii Nelineinykh Kolebanii* (Moscow: Gos. Izdat Tekhniko-Teor. Lit.)

Born N 1960 *The Mechanics of the Atom* (New York: Ungar)

Burshtein E L and Solovev L S 1961 *Dokl Akad. Nauk.* **139** 855

Coffey T P 1969 *J. Math. Phys* **10** 426

Coffey T P and Ford G F 1969 *J. Math. Phys.* **10** 998

Fedortshenko A M 1957 *Ukr Mat. Zh.* **9** 220

Fues E 1927 *Handb. Phys* **5** 131 (Berlin: Springer-Verlag)

Giacaglia G E O 1972 *Perturbation Methods in Non-Linear Systems* (New York: Springer-Verlag)

Hori G 1966 *Publ. Astr. Soc. Japan* **18** 287

Mitropolskii Yu A 1971 *Metod Usrednenia v Nelineinoi Mekhanike* (Kiev: Naukova Dumka)

Sedláček Z 1974 *Phys. Lett.* **48A** 287

—— 1975 *J. Phys* A: *Math. Gen.* **8** 1067–74

Stern D P 1970 *J. Math. Phys.* **11** 2776

Sturrock P A 1964 *Advanced Plasma Theory* (*Proc. Int. School of Physics, Varenna,* 1962) (New York and London: Academic Press) p 180